

## SOME AGGREGATION OPERATORS FOR BIPOLAR-VALUED HESITANT FUZZY INFORMATION BASED ON EINSTEIN OPERATIONAL LAWS

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### ABSTRACT

*This article is based on Einstein operations for bipolar-valued hesitant fuzzy sets (BVHFSs). We extend the concept of Einstein operators for BVHFSs by defining bipolar-valued hesitant fuzzy Einstein weighted averaging (BPVHFEWA) operators and weighted geometric (BPVHFEWG) operators. Similarly, we define ordered weighted averaging operators and hybrid operators i.e. BVHFEOWA operators, BVHFEOWG operators, BVHFEHA operator and BVHFEHG operator. Further, these operators are applied in decision making (DM) problems.*

**KEYWORDS:** *Bipolar-valued hesitant fuzzy sets. Einstein operations. Aggregation operators. Decision making.*

### INTRODUCTION

In the field of fuzzy mathematics decision making (DM) has a great importance. In fact, DM plays an important role in every walk of our daily life. At different stages of his life any person would have to take decision about his career. DM problem are so complex that ordinary mathematical techniques are not be able to deal with it.

(Zadeh,1965) gave rise to a new concept known as fuzzy set (FS). FSs have been discovered to deal with any kind of uncertainties and it plays a successful role in dealing with any kind of fuzziness. Soon after the discovery of FSs different new advance extensions of FSs are developed. Some famous extensions of FSs are hesitant fuzzy sets (HFSs) (Torra et al., 2009 &2010), intuitionistic fuzzy sets (IFSs), inter-valued fuzzy sets (IVFSs) (Dubois and Prade, 2010), bipolar-valued fuzzy sets (BVFSs) (Lee, 2000). Almost all of these extensions of FSs are somehow related to DM process. In fact in DM we need to look at the basic properties of concerned FS. For some related work one may refer to (Chauhan and Vaish, 2014; De et al., 2000; Farhadinia, 2013; Herrera-Viedma et al., 2007; Hong and Choi, 2000; Klutho, 2013; Li, 2005; Liu and Wang 2007; Abdullah et al., 2014).

Among different extensions of FSs, the theory of HFSs is of great interest. It was presented by Torra and Narukawa (2009) and it was indeed a unique idea of its kind that HFSs have affiliation degrees in the form of finite elements. This was such an interesting theory

that most of the scientist took great interest in HFSs and soon a score function was defined in order to measure a HFS. (Chen, 2011; Chen et al., 2013; Chen and Xu, 2014) presented a function for the deviation degree HFSs. (Xia et al., (2013) used HFSs in DM process. For related work one may refer to (Liu et al., 2017; Neves and Livet 2008; Shabir and Israr 2008; Torra and Narukawa 2009; Torra, 2010; Xia and Xu, 2011; Xia et al., 2013; Xu, 2004; Xu and Yager, 2006; Xu, 2007; Xu and Jian, 2007; Ye, 2007; Ye, 2009).

Mahmood et al., (2016), presented a new theory known as BVHFS. BVHFS is a new enhanced extension of FSs developed by combining BVFSs and HFSs. Membership functions of BVHFSs defined in terms of finite values from  $[-1,1]$ . The positive membership function is a finite set whose values are taken from  $[0,1]$  conveying the level of satisfaction of an element. The negative membership function is a finite set whose values are from  $[-1,0]$  conveying the counter satisfaction level of an element. Mahmood et al., (2016), developed some basic theory for BVHFS and proved useful results of interest. Some aggregation operators were also defined for BVHFS and then applied in DM process in Mahmood et al., (2017). For more work (Mahmood et al., 2013; Mahmood et al., 2016a; Mahmood et al., 2017, Mahmood et al., 2016b).

The concept of Einstein operations was laid down by (Wang and Liu, 2011) at first. They defined Einstein operations for IFSs and they also developed theory of aggregation operators based on Einstein operations. They also defined Einstein T-norm and T-conorm and developed the theory of Einstein operations. Motivated

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by the concepts of (Wang and Liu, 2011) soon, Dejian Yu take the concept of Einstein operations in the field of HFSs and define some very basic operation for HFSs based on Einstein operations. Yu (2014) also defined several aggregation operators of HFSs based on Einstein operations and then apply these concepts to a MADM problem.

We in this treatise, define Einstein operations for BVHFS. We get inspired by Yu (2014) defined some aggregation operators for BVHFS based on Einstein operation. We prove some interesting results and support our ideas with the help of few examples. Finally we apply the newly defined aggregation operators to MADM problem and find these operations very beneficial in DM process.

Starting with introduction, this article has five sections. In section two, we present some previous concepts related to our work which includes HFSs, IFSs, BVHFSs and some defined Einstein operations. In section three we define Einstein operation for BVHFSs and present an example in support of our newly defined operations. Section four deals with aggregation operators for BVHFSs. We also proved some useful results in this section along with few examples. The last section consists of DM process briefly and a MADM problem solved by using aggregation operators defined in section four. Finally, we conclude our article by adding some concluding remarks.

## PRELIMINARIES

This section consists of some previous concepts to support our new work. It contains concepts related to BVHFSs and their operations along with Einstein operations for HFSs.

### Definition 1: (Atanassov, 1986)

An IFS on a set  $X$  is of the form where  $\mu_a : X \rightarrow [0,1] \forall x \in X$  is known as a membership function and  $\nu_a : X \rightarrow [0,1] \forall x \in X$  is known as a non-membership function provided that  $0 \leq \mu_a(x) + \nu_a(x) \leq 1 \forall x \in X$ . Here  $\mu_a(x)$  and  $\nu_a(x)$  represents the degree of membership and non-membership of element  $x \in X$ . For our ease  $a = (\mu_a, \nu_a)$  is called an intuitionistic fuzzy number (IFN).

Wang and Liu bring forth some new operations for

IFVs based on Einstein operations which are described below:

### Definition 2: (Wang and Liu 2011)

Let  $\alpha, \alpha_1, \alpha_2$  be three IFVs then we define some Einstein operations on  $\alpha, \alpha_1, \alpha_2$  as follows:

1.  $\alpha_1 \oplus \alpha_2 = \left( \frac{\mu_{\alpha_1} + \nu_{\alpha_1}}{1 + \mu_{\alpha_1} \cdot \nu_{\alpha_1}}, \frac{\mu_{\alpha_2} \cdot \nu_{\alpha_2}}{1 + (1 - \mu_{\alpha_2})(1 - \nu_{\alpha_2})} \right)$
2.  $\alpha_1 \otimes \alpha_2 = \left( \frac{\mu_{\alpha_1} \cdot \nu_{\alpha_1}}{1 + (1 - \mu_{\alpha_1})(1 - \nu_{\alpha_1})}, \frac{\mu_{\alpha_2} + \nu_{\alpha_2}}{1 + \mu_{\alpha_2} \cdot \nu_{\alpha_2}} \right)$
3.  $\rho \cdot \alpha = \left( \frac{(1 + \mu_a)^{\rho'} - (1 - \mu_a)^{\rho'}}{(1 + \mu_a)^{\rho'} + (1 - \mu_a)^{\rho'}}, \frac{2\mu_a^{\rho'}}{(2 - \mu_a)^{\rho'} + \mu_a^{\rho'}} \right) \rho'$  is any constant
4.  $\alpha^{\rho} = \left( \frac{2\mu_a^{\rho'}}{(2 - \mu_a)^{\rho'} + \mu_a^{\rho'}}, \frac{(1 + \mu_a)^{\rho'} - (1 - \mu_a)^{\rho'}}{(1 + \mu_a)^{\rho'} + (1 - \mu_a)^{\rho'}} \right) \rho'$  is any constant

### Definition 3: (Torra and Narukawa, 2009; Torra, 2010)

Let  $X$  be any set. Then a HFS on  $X$  in the form of a function  $H$  that when applied to  $X$  gives us few values in  $[0,1]$ . A HFS on  $X$  is denoted as  $H = \{x, h(x) \mid \forall x \in X\}$  where  $h(x)$  is a finite set whose values are form  $[0,1]$ . The set of all HFSs is expressed by  $H'$ . Here  $h(x)$  is known as HFE.

Xia and Xu developed some operations on HFSs which are given as follows:

1.  $H^{\rho'}(x) = \{m^{\rho'} : m \in H(x)\}$
2.  $\rho'H(x) = \{1 - (1 - m)^{\rho'} : m \in H(x)\}$

### Definition 4: (Yu, 2014)

According to Dejian Yu, Einstein operations for three HFEs are defined as follows:

1.  $H_1 \oplus H_2 = \bigcup_{m_1 \in H_1, m_2 \in H_2} \left\{ \frac{m_1 + m_2}{1 + m_1 m_2} \right\}$  is any constant
2.  $H_1 \otimes H_2 = \bigcup_{m_1 \in H_1, m_2 \in H_2} \left\{ \frac{m_1 m_2}{1 + (1 - m_1)(1 - m_2)} \right\}$  is any constant
3.  $\rho'H = \bigcup_{m \in H} \left\{ \frac{(1 + m)^{\rho'} - (1 - m)^{\rho'}}{(1 + m)^{\rho'} + (1 - m)^{\rho'}} \right\} \rho'$  is any constant
4.  $H^{\rho} = \bigcup_{m \in H} \left\{ \frac{2m^{\rho'}}{(2 - m)^{\rho'} + m^{\rho'}} \right\} \rho'$  is any constant

### Definition 5: (Mahmood et al., 2016a)

For any set  $X$ , the BVHFS  $\mathfrak{B}$  on some domain of  $X$

is denoted and defined by:

$$\mathfrak{B} = \{x, (H^+(x), H^-(x)) : x \in X\}$$

Where  $H^+:X \rightarrow [0,1]$  is a finite set whose values are from  $[0,1]$  conveying the satisfaction level of “x” to BVHFS  $\mathfrak{B}$  and  $H^-:X \rightarrow [-1,0]$  is a finite set whose values are from  $[-1,0]$  conveying the dissatisfaction level of “x” to BVHFS  $\mathfrak{B}$ . Here  $H = \{H^+(x), H^-(x)\}$  is a BVHFE. The set of all BVHFEs is denoted by  $\Phi$ .

Consider two BVHFSs.

$$\mathfrak{A} = \{x, (H^+_{\mathfrak{A}}(x), H^-_{\mathfrak{A}}(x)) : x \in X\}$$

$$\mathfrak{B} = \{x, (H^+_{\mathfrak{B}}(x), H^-_{\mathfrak{B}}(x)) : x \in X\}$$

Some operations for BVHFSs are as follows:

$$(\mathfrak{A} \oplus \mathfrak{B})(x) = \{m_1 + m_2 - m_1 m_2 : m_1 \in H^+_{\mathfrak{A}}(x), m_2 \in H^+_{\mathfrak{B}}(x), \\ - (m_1 m_2) : m_1 \in H^-_{\mathfrak{A}}(x), m_2 \in H^-_{\mathfrak{B}}(x)\}$$

$$(\mathfrak{A} \otimes \mathfrak{B})(x) = \{m_1 m_2 : m_1 \in H^+_{\mathfrak{A}}(x), m_2 \in H^+_{\mathfrak{B}}(x), \\ - (m_1 - m_2 - m_1 m_2) : m_1 \in H^-_{\mathfrak{A}}(x), m_2 \in H^-_{\mathfrak{B}}(x)\}$$

For any  $\rho > 0$

**Definition 6:** (Mahmood et al., 2016a)

Let  $H = \{H^+, H^-\}$  be a BVHFE. Then score function  $S$  of  $H$  is defined as:

$$S(H) = \frac{1}{\ell_H} (\xi_H^+ + \xi_H^-)$$

Where  $\xi_H^+$  is the total of positive membership grades and  $\xi_H^-$  is the total of negative membership grades and  $S(H) \in [-1,1]$ .

**Remark 1:** (Mahmood et al., 2016a)

Length  $H^+$  and  $H^-$  of are not necessarily equal.

For two BVHFEs  $H_1$  and  $H_2$  if

$S(H_1) < S(H_2)$ , then  $H_1$  is smaller than  $H_2$  i.e.  $H_1 < H_2$ .

$S(H_1) > S(H_2)$ , then  $H_1$  is bigger than  $H_2$  i.e.  $H_1 > H_2$ .

$S(H_1) = S(H_2)$ , then  $H_1$  is indifferent (similar) to  $H_2$  denoted by  $H_1 \sim H_2$ .

Yu, (2014), defined Einstein operation for HFSs. He also defines several aggregation operators for HFSs. Enhancing the same concept, we developed the same operation for BVHFSs. We are also successful in defining several aggregation operators for BVHFSs and then we apply these operators to a MADM problem and get quite useful result.

Einstein Operations for Bipolar-Valued Hesitant Fuzzy Sets

**Definition 7:**

Let  $H, H_1, H_2$  three BVHFEs then Einstein operations are defined as:

1.  $H_1 \oplus H_2 = \bigcup_{m_1 \in H_1, m_2 \in H_2} \left\{ \frac{m_1 + m_2}{1 + m_1 m_2}, \left\{ \frac{-(-m_1)(-m_2)}{-1 - (-1 - m_1)(-1 - m_2)} \right\} \right\}$
2.  $H_1 \otimes H_2 = \bigcup_{m_1 \in H_1, m_2 \in H_2} \left\{ \frac{m_1 m_2}{1 + (1 - m_1)(1 - m_2)}, \left\{ \frac{-(-m_1 - m_2)}{-1 - (-1 - m_1)(-1 - m_2)} \right\} \right\}$
3.  $\rho^H = \bigcup_{m \in H} \left\{ \frac{\left( \frac{(1+m)^{\rho} - (1-m)^{\rho}}{(1+m)^{\rho} + (1-m)^{\rho}} \right)}{\left( \frac{(-2-m)^{\rho} - (-(-m))^{\rho}}{(-2-m)^{\rho} - (-(-m))^{\rho}} \right)} \right\} \rho$  is any constant
4.  $H^{\rho} = \bigcup_{m \in H} \left\{ \frac{2m^{\rho}}{(2-m)^{\rho} + m^{\rho}}, \left\{ \frac{-((-1-m))^{\rho} - (-(-1-m))^{\rho}}{(-2-m)^{\rho} - (-(-m))^{\rho}} \right\} \right\} \rho$  is any constant

**Example 1:**

Let  $H = \{\{0.1, 0.2\}, \{-0.3, -0.2\}\}$ ,  $H_1 = \{\{0.5, 0.6\}, \{-0.2, -0.1\}\}$

$H_2 = \{\{0.9, 0.8\}, \{-0.5, -0.4\}\}$  then

$$H_1 \oplus H_2 = \{\{0.965517, 0.928571, 0.974026, 0.945946\}, \{-0.07143, -0.05405, -0.03448, -0.02597\}\}$$

$$H_1 \otimes H_2 = \{\{0.428571, 0.363636, 0.519231, 0.444444\}, \{-0.63636, -0.55556, -0.57143, -0.48077\}\}$$

When  $\rho = 2$  then

$$\rho H = \{\{0.19802, 0.384615\}, \{-0.06429, -0.025\}\}$$

$$H^{\rho} = \{\{0.004535, 0.016529\}, \{-0.42857, -0.25\}\}$$

**Aggregation Operators**

Based on Einstein operational law, some aggregation operators for BPVHF are developed in this section. The proposed operators are briefly explored with the examples. In our further study, the following should be kept in mind:

$w_i^T = (w_1, w_2, \dots, w_n)^T$  denotes the weight vectors.

$w' = (w'_1, w'_2, \dots, w'_n)^T$  denotes aggregation-associated weight vector.

$H_i$  represents some BVHFEs.

$(i, j, k = 1, 2, 3, 4, \dots, n)$  be some set of indices.

### Definition 8:

For some BVHFEs  $H_i$  and their weight vectors  $w'_i$  such that  $w'_i \in [0, 1]$  and  $\sum_{i=1}^n w'_i = 1$ , then BVHFEWA operator is a function  $\Psi^n \rightarrow \Psi$  such that  $BVHFEWA(H_1, H_2, \dots, H_n) = \bigoplus_{i=1}^n (w'_i H_i)$

if we choose the weight vector  $w' = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$  then BVHFEWA operator becomes BVHFEA operator i.e.

$$BVHFEA(H_1, H_2, \dots, H_n) = \bigoplus_{i=1}^n \left( \frac{1}{n} H_i \right)$$

Using the above definitions, by applying induction technique on 'n' some results are proved.

### Theorem 1:

For some BVHFEs  $H_i$ . Their aggregated value determined by using BVHFEWA operator is a BVHFE and

$$BVHFEWA(H_1, H_2, \dots, H_n) = \bigcup_{n_i \in H_i} \left\{ \left( \frac{\prod_{j=1}^n (1+m_j)^{w'_j} - \prod_{j=1}^n (1-m_j)^{w'_j}}{\prod_{j=1}^n (1+m_j)^{w'_j} + \prod_{j=1}^n (1-m_j)^{w'_j}} \right), \left( \frac{-2 \prod_{j=1}^n (-m_j)^{w'_j}}{\prod_{j=1}^n (-(-2-m_j))^{w'_j} - (-2 \prod_{j=1}^n (-m_j)^{w'_j})} \right) \right\}$$

Proof:

The first part of this theorem is obvious. To prove the second part, we need to use induction. First, the result is proved true for  $n=2$ .

When  $n=2$

$$w'_1 H_1 = \left\{ \left( \frac{(1+m_1)^{w'_1} - (1-m_1)^{w'_1}}{(1+m_1)^{w'_1} + (1-m_1)^{w'_1}} \right), \left( \frac{-2(-m_1)^{w'_1}}{(-(-2-m_1))^{w'_1} - (-(-m_1)^{w'_1})} \right) \right\}$$

$$w'_2 H_2 = \left\{ \left( \frac{(1+m_2)^{w'_2} - (1-m_2)^{w'_2}}{(1+m_2)^{w'_2} + (1-m_2)^{w'_2}} \right), \left( \frac{-2(-m_2)^{w'_2}}{(-(-2-m_2))^{w'_2} - (-(-m_2)^{w'_2})} \right) \right\}$$

Now  $w'_1 H_1 \oplus w'_2 H_2$

$$= \left\{ \left\{ \left( \frac{(1+m_1)^{w'_1} - (1-m_1)^{w'_1}}{(1+m_1)^{w'_1} + (1-m_1)^{w'_1}} \right), \left( \frac{-2(-m_1)^{w'_1}}{(-(-2-m_1))^{w'_1} - (-(-m_1)^{w'_1})} \right) \right\}, \left\{ \left( \frac{(1+m_2)^{w'_2} - (1-m_2)^{w'_2}}{(1+m_2)^{w'_2} + (1-m_2)^{w'_2}} \right), \left( \frac{-2(-m_2)^{w'_2}}{(-(-2-m_2))^{w'_2} - (-(-m_2)^{w'_2})} \right) \right\} \right\}$$

$$= \left\{ \left\{ \left( \frac{(1+m_1)^{w'_1} - (1-m_1)^{w'_1}}{(1+m_1)^{w'_1} + (1-m_1)^{w'_1}} \right), \left( \frac{(1+m_2)^{w'_2} - (1-m_2)^{w'_2}}{(1+m_2)^{w'_2} + (1-m_2)^{w'_2}} \right) \right\}, \left( \frac{-2(-m_1)^{w'_1} - 2(-m_2)^{w'_2}}{(-(-2-m_1))^{w'_1} - (-(-m_1)^{w'_1})} \right) \right\}$$

$$= \left\{ \left( \frac{-2(-m_1)^{w'_1} - 2(-m_2)^{w'_2}}{(-(-2-m_1))^{w'_1} - (-(-m_1)^{w'_1})} \right), \left( \frac{-2(-m_2)^{w'_2}}{(-(-2-m_2))^{w'_2} - (-(-m_2)^{w'_2})} \right) \right\}$$

$$= \left\{ \left( \frac{-2 \prod_{j=1}^2 (-m_j)^{w'_j}}{\prod_{j=1}^2 (-(-2-m_j))^{w'_j} - (-2 \prod_{j=1}^2 (-m_j)^{w'_j})} \right), \left( \frac{-2 \prod_{j=1}^2 (-m_j)^{w'_j}}{\prod_{j=1}^2 (-(-2-m_j))^{w'_j} - (-2 \prod_{j=1}^2 (-m_j)^{w'_j})} \right) \right\}$$

Now we suppose that the given result holds for  $n=k$

$$\bigoplus_{j=1}^k (w'_j H_j) = \left\{ \left( \frac{\prod_{j=1}^k (1+m_j)^{w'_j} - \prod_{j=1}^k (1-m_j)^{w'_j}}{\prod_{j=1}^k (1+m_j)^{w'_j} + \prod_{j=1}^k (1-m_j)^{w'_j}} \right), \left( \frac{-2 \prod_{j=1}^k (-m_j)^{w'_j}}{\prod_{j=1}^k (-(-2-m_j))^{w'_j} - (-2 \prod_{j=1}^k (-m_j)^{w'_j})} \right) \right\}$$

Now we prove that the result holds true for  $n=k+1$

$$\bigoplus_{j=1}^{k+1} (w'_j H_j) = \bigoplus_{j=1}^k (w'_j H_j) \oplus w'_{k+1} H_{k+1}$$

$$= \left\{ \left( \frac{\prod_{j=1}^k (1+m_j)^{w'_j} - \prod_{j=1}^k (1-m_j)^{w'_j}}{\prod_{j=1}^k (1+m_j)^{w'_j} + \prod_{j=1}^k (1-m_j)^{w'_j}} \right), \left( \frac{-2 \prod_{j=1}^k (-m_j)^{w'_j}}{\prod_{j=1}^k (-(-2-m_j))^{w'_j} - (-2 \prod_{j=1}^k (-m_j)^{w'_j})} \right) \right\} \oplus$$

$$\left\{ \left( \frac{(1+m_{k+1})^{w'_{k+1}} - (1-m_{k+1})^{w'_{k+1}}}{(1+m_{k+1})^{w'_{k+1}} + (1-m_{k+1})^{w'_{k+1}}} \right), \left( \frac{-2(-m_{k+1})^{w'_{k+1}}}{(-(-2-m_{k+1}))^{w'_{k+1}} - (-2(-m_{k+1})^{w'_{k+1}})} \right) \right\}$$

$$= \left\{ \left( \frac{\prod_{j=1}^{k+1} (1+m_j)^{w'_j} - \prod_{j=1}^{k+1} (1-m_j)^{w'_j}}{\prod_{j=1}^{k+1} (1+m_j)^{w'_j} + \prod_{j=1}^{k+1} (1-m_j)^{w'_j}} \right), \left( \frac{-2 \prod_{j=1}^{k+1} (-m_j)^{w'_j}}{\prod_{j=1}^{k+1} (-(-2-m_j))^{w'_j} - (-2 \prod_{j=1}^{k+1} (-m_j)^{w'_j})} \right) \right\}$$

Hence the result holds true for  $n=k+1$  so the result holds true for all  $n$  and we have

$$BVHFEWA(H_1, H_2, \dots, H_n) = \left\{ \left( \frac{\prod_{j=1}^n (1+m_j)^{w'_j} - \prod_{j=1}^n (1-m_j)^{w'_j}}{\prod_{j=1}^n (1+m_j)^{w'_j} + \prod_{j=1}^n (1-m_j)^{w'_j}} \right), \left( \frac{-2 \prod_{j=1}^n (-m_j)^{w'_j}}{\prod_{j=1}^n (-(-2-m_j))^{w'_j} - (-2 \prod_{j=1}^n (-m_j)^{w'_j})} \right) \right\}$$

### Definition 9:

For some BVHFEs  $H_i$  and their weight vectors  $w'_i$  such that  $w'_i \in [0, 1]$  and  $\sum_{i=1}^n w'_i = 1$ . The BVHFEWG operator is a function  $\Psi^n \rightarrow \Psi$  such that

$$BVHFEWG(H_1, H_2, \dots, H_n) = \bigotimes_{i=1}^n (H_i)^{w'_i}$$

If we choose the weight vector,  $w' = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$  then BVHFEWG operator becomes BVHFEF operator i.e.

$$BVHFEG(H_1, H_2, \dots, H_n) = \left( \bigotimes_{i=1}^n (H_i) \right)^{\frac{1}{n}}$$

Theorem 2:

For some BVHFEs  $H_i$ . Their aggregate calculated by using BVHFEWG operator is a BVHFE and

$$BVHFEWG(H_1, H_2, \dots, H_n) = \left[ \frac{2 \prod_{j=1}^n (m_j)^{w_j}}{\prod_{j=1}^n (2 - m_j)^{w_j} - \prod_{j=1}^n (m_j)^{w_j}} \right] \left[ \frac{-\left( \prod_{j=1}^n (-(-1 - (m_j)))^{w_j} \right) - \prod_{j=1}^n (-(-1 - (m_j)))^{w_j}}{\prod_{j=1}^n (-(-2 - (m_j)))^{w_j} - \left( -\left( \prod_{j=1}^n (-(-m_j))^{w_j} \right) \right)} \right]$$

This result can be proved in analogous way by principle of mathematical induction as done in previous result.

Example 2:

Let  $H_1 = \{0.5, 0.6\}, \{-0.2, -0.1\}$  and  $H_2 = \{0.9, 0.8\}, \{-0.5, -0.4\}$  be two BVHFEs and let  $w = (0.35, 0.65)^{\frac{1}{2}}$  then we calculate their aggregated value by BVHFEWA operator and BVHFEWG operator as follows:

$$BVHFEWA(H_1, H_2) = \left[ \frac{\prod_{j=1}^2 (1 + m_j)^{w_j} - \prod_{j=1}^2 (1 - m_j)^{w_j}}{\prod_{j=1}^2 (1 + m_j)^{w_j} - \prod_{j=1}^2 (1 - m_j)^{w_j}} \right] \left[ \frac{-2 \prod_{j=1}^2 (-m_j)^{w_j}}{\prod_{j=1}^2 (-(-2 - m_j))^{w_j} - (-2 \prod_{j=1}^2 (-m_j)^{w_j})} \right]$$

$$= \{0.817489, 0.719378, 0.833516, 0.742801\}, \{-0.09103, -0.06262, -0.06475, -0.04359\}$$

$$BVHFEWG(H_1, H_2) = \left[ \frac{2 \prod_{j=1}^2 (m_j)^{w_j}}{\prod_{j=1}^2 (2 - m_j)^{w_j} - \prod_{j=1}^2 (m_j)^{w_j}} \right] \left[ \frac{-\left( \prod_{j=1}^2 (-(-1 - (m_j)))^{w_j} \right) - \prod_{j=1}^2 (-(-1 - (m_j)))^{w_j}}{\prod_{j=1}^2 (-(-2 - (m_j)))^{w_j} - \left( -\left( \prod_{j=1}^2 (-(-m_j))^{w_j} \right) \right)} \right]$$

$$= \{0.748068, 0.686851, 0.789691, 0.727046\}, \{-0.64554, -0.48977, -0.54397, -0.40964\}$$

**Definition 10:**

For some BVHFEs  $H_i$ , ( $H_{\sigma(i)}$  is the  $i^{\text{th}}$  largest among them) and their weight vectors  $w_i^T$  such that  $w_i \in [0, 1]$  and  $\sum_{i=1}^n w_i = 1$ .

1. A BVHFEOWA operator is a function  $BVHFEOWA: \Psi^n \rightarrow \Psi$ , such that

$$BVHFEOWA(H_1, H_2, \dots, H_n) = \bigoplus_{i=1}^n (w_i^T H_{\sigma(i)}) = w_1^T H_{\sigma(1)} + w_2^T H_{\sigma(2)} + \dots + w_n^T H_{\sigma(n)}$$

2. A BVHFEOWG operator is a function  $BVHFEOWG: \Psi^n \rightarrow \Psi$ , such that

$$BVHFEOWG(H_1, H_2, \dots, H_n) = \bigotimes_{i=1}^n (H_{\sigma(i)})^{w_i} = H_{\sigma(1)}^{w_1} \times H_{\sigma(2)}^{w_2} \times H_{\sigma(3)}^{w_3} \times \dots \times H_{\sigma(n)}^{w_n}$$

**Theorem 3:**

For some BVHFEs  $H_i$ , their aggregate calculated by using BVHFEOWA operator or BVHFEOWG operator is a BVHFE and

$$1. BVHFEOWA(H_1, H_2, \dots, H_n) = \bigcup_{m_{\sigma(i)} \in H_{\sigma(i)}} \left\{ \left[ \frac{\prod_{j=1}^n (1 + m_{\sigma(j)})^{w_j} - \prod_{j=1}^n (1 - m_{\sigma(j)})^{w_j}}{\prod_{j=1}^n (1 + m_{\sigma(j)})^{w_j} - \prod_{j=1}^n (1 - m_{\sigma(j)})^{w_j}} \right] \left[ \frac{-2 \prod_{j=1}^n (-m_{\sigma(j)})^{w_j}}{\prod_{j=1}^n (-(-2 - m_{\sigma(j)}))^{w_j} - (-2 \prod_{j=1}^n (-m_{\sigma(j)})^{w_j})} \right] \right\}$$

$$2. BVHFEOWG(H_1, H_2, \dots, H_n) = \bigcup_{m_{\sigma(i)} \in H_{\sigma(i)}} \left\{ \left[ \frac{2 \prod_{j=1}^n (m_{\sigma(j)})^{w_j}}{\prod_{j=1}^n (2 - m_{\sigma(j)})^{w_j} - \prod_{j=1}^n (m_{\sigma(j)})^{w_j}} \right] \left[ \frac{-\left( \prod_{j=1}^n (-(-1 - (m_{\sigma(j)})))^{w_j} \right) - \prod_{j=1}^n (-(-1 - (m_{\sigma(j)})))^{w_j}}{\prod_{j=1}^n (-(-2 - (m_{\sigma(j)})))^{w_j} - \left( -\left( \prod_{j=1}^n (-(-m_{\sigma(j)}))^{w_j} \right) \right)} \right] \right\}$$

**Example 3:**

Let  $H_1 = \{0.1, 0.2\}, \{-0.3, -0.2\}$ ,  $H_2 = \{0.5, 0.6\}, \{-0.2, -0.1\}$  and  $H_3 = \{0.9, 0.8\}, \{-0.2, -0.1\}$  be three BVHFEs and let  $w = (0.3, 0.5, 0.2)^{\frac{1}{3}}$  be the aggregation- associated weight vector. Then

$$S(H_1) = \frac{1}{\ell_{H_1}} (\xi_{H_1}^+ + \xi_{H_1}^-) = \frac{1}{2} (0.1 + 0.2 + (-0.3) + (-0.2)) = -0.1$$

$$S(H_2) = \frac{1}{\ell_{H_2}} (\xi_{H_2}^+ + \xi_{H_2}^-) = \frac{1}{2} (0.5 + 0.6 + (-0.2) + (-0.1)) = 0.8$$

$$S(H_3) = \frac{1}{\ell_{H_3}} (\xi_{H_3}^+ + \xi_{H_3}^-) = \frac{1}{2} (0.9 + 0.8 + (-0.2) + (-0.1)) = 0.7$$

Clearly as

$$S(H_1) < S(H_3) < S(H_2)$$

So

$$H_{\sigma(1)} = H_2 = \{0.5, 0.6\}, \{-0.2, -0.1\}, H_{\sigma(2)} = H_3 = \{0.9, 0.8\}, \{-0.2, -0.1\} \text{ and } H_{\sigma(3)} = H_1 = \{0.1, 0.2\}, \{-0.3, -0.2\}$$

Now

$$BVHFEOWA(H_1, H_2, H_3) = \left\{ \left[ \frac{0.7264, 0.7359, 0.6256, 0.6379}{0.7461, 0.7551, 0.6512, 0.6628} \right], \left[ \frac{-0.2173, -0.2, -0.1548, -0.1421}{-0.1775, -0.1631, -0.1257, -0.1152} \right] \right\}$$

$$BVHFEOWG(H_1, H_2, H_3) = \left\{ \left[ \frac{0.5305, 0.5908, 0.4916, 0.5491}{0.5604, 0.6226, 0.5201, 0.5796} \right], \left[ \frac{-0.2819, -0.2500, -0.2038, -0.1759}{-0.2334, -0.2041, -0.1624, -0.1361} \right] \right\}$$

Here we see that BVHFEWA operator and BVHFEWG operators weight the BVHF argument but it gives no importance to the order of argument while BVHFEOWA operator and BVHFEOWG operator give weight to the ordered position of the statement and ignore its worth. Enhancing these concepts, we are going to introduce the concept of hybrid operators that weight the ordered

position of statement along with its importance.

**Definition 11:**

For some BVHFEs  $H_i$  and their weight vectors  $w_i^T$  such that  $w_i \in [0,1]$  and  $\sum_{j=1}^n w_j = 1$ . Here 'n' is the balancing coefficient and  $w_i^T$  be the weight vector for aggregation process of  $H_i$  with  $w_i \in [0,1]$  and  $\sum_{j=1}^n w_j = 1$ .

1. The BVHFEHA operator is a mapping  $BVHFEHA: \Psi^n \rightarrow \Psi$  such that

$$BPVHFEHA(\dot{H}_1, \dot{H}_2, \dots, \dot{H}_n) = \bigoplus_{i=1}^n \left( w_i \dot{H}_{\sigma(i)} \right) = w_1 \dot{H}_{\sigma(1)} + w_2 \dot{H}_{\sigma(2)} + \dots + w_n \dot{H}_{\sigma(n)}$$

where  $\dot{H}_{\sigma(i)}$  is the  $i^{\text{th}}$  largest of  $H = n w_k H_k$  ( $k=1,2,3,\dots,n$ )

2. e BVHFEHG operator is a mapping  $BVHFEHG: \Psi^n \rightarrow \Psi$  such that

$$BVHFEHG(\ddot{H}_1, \ddot{H}_2, \dots, \ddot{H}_n) = \bigotimes_{i=1}^n \left( \ddot{H}_{\sigma(i)} \right)^{w_i} = \ddot{H}_{\sigma(1)}^{w_1} \times \ddot{H}_{\sigma(2)}^{w_2} \times \dots \times \ddot{H}_{\sigma(n)}^{w_n}$$

where  $\ddot{H}_{\sigma(i)}$  is the  $i^{\text{th}}$  largest of  $H = H_k^{n w_k}$  ( $k=1,2,3,\dots,n$ )

**Theorem 4:**

For some BVHFEs  $H_i$ , their aggregate calculated by using BVHFEHA operator or BVHFEHG operator is a BVHFE and

$$\begin{aligned} & 1. BPVHFEHA(\dot{H}_1, \dot{H}_2, \dots, \dot{H}_n) \\ &= \bigcup_{\substack{m_{\sigma(i)} \in \dot{H}_{\sigma(i)} \\ \left[ \frac{\prod_{j=1}^n (1 + \dot{m}_{\sigma(j)})^{w_j} - \prod_{j=1}^n (1 - \dot{m}_{\sigma(j)})^{w_j}}{\prod_{j=1}^n (1 + \dot{m}_{\sigma(j)})^{w_j} - \prod_{j=1}^n (1 - \dot{m}_{\sigma(j)})^{w_j}} \right] \left[ \frac{-2 \prod_{j=1}^n (-\dot{m}_{\sigma(j)})^{w_j}}{\prod_{j=1}^n (-2 - \dot{m}_{\sigma(j)})^{w_j} - (-2 \prod_{j=1}^n (-\dot{m}_{\sigma(j)})^{w_j})} \right]} \\ & 2. BVHFEHG(\ddot{H}_1, \ddot{H}_2, \dots, \ddot{H}_n) \\ &= \bigcup_{\substack{m_{\sigma(i)} \in \ddot{H}_{\sigma(i)} \\ \left[ \frac{2 \prod_{j=1}^n (\ddot{m}_{\sigma(j)})^{w_j}}{\prod_{j=1}^n (2 - \ddot{m}_{\sigma(j)})^{w_j} - \prod_{j=1}^n (\ddot{m}_{\sigma(j)})^{w_j}} \right] \left[ \frac{-\left( \prod_{j=1}^n (-1 - (\ddot{m}_{\sigma(j)})^{w_j}) \right) - \prod_{j=1}^n (-1 - (\ddot{m}_{\sigma(j)})^{w_j})}{\prod_{j=1}^n (-2 - (\ddot{m}_{\sigma(j)})^{w_j}) - \left( -\prod_{j=1}^n (-\ddot{m}_{\sigma(j)})^{w_j} \right)} \right]} \end{aligned}$$

Example 4:

Let  $H_1 = \{ \{0.1, 0.2\}, \{-0.3, -0.2\} \}$ ,  $H_2 = \{ \{0.5, 0.6\}, \{-0.2, -0.1\} \}$  and  $H_3 = \{ \{0.9, 0.8\}, \{-0.2, -0.1\} \}$  be three BVHFEs and  $w = (0.15, 0.2, 0.65)$  be their weight vector and let  $w = (0.3, 0.5, 0.2)^T$  be the aggregation- associated vector. Then

$$\begin{aligned} \dot{H}_1 &= \left\{ \left\{ 1 - (1 - 0.1)^{2 \times 0.15}, 1 - (1 - 0.2)^{2 \times 0.15} \right\}, \left\{ -(-(-0.3))^{2 \times 0.15}, -(-(-0.2))^{2 \times 0.15} \right\} \right\} \\ \dot{H}_1 &= \{ \{0.031114, 0.064752\}, \{-0.69685, -0.61703\} \} \end{aligned}$$

$$\begin{aligned} \dot{H}_2 &= \left\{ \left\{ 1 - (1 - 0.5)^{2 \times 0.2}, 1 - (1 - 0.6)^{2 \times 0.2} \right\}, \left\{ -(-(-0.2))^{2 \times 0.2}, -(-(-0.1))^{2 \times 0.2} \right\} \right\} \\ \dot{H}_2 &= \{ \{0.242142, 0.306855\}, \{-0.52531, -0.39811\} \} \\ \dot{H}_3 &= \left\{ \left\{ 1 - (1 - 0.9)^{2 \times 0.65}, 1 - (1 - 0.8)^{2 \times 0.65} \right\}, \left\{ -(-(-0.2))^{2 \times 0.65}, -(-(-0.1))^{2 \times 0.65} \right\} \right\} \\ \dot{H}_3 &= \{ \{0.949881, 0.876593\}, \{-0.12341, -0.05012\} \} \end{aligned}$$

The score values are:

$$\begin{aligned} S(\dot{H}_1) &= \frac{1}{\ell_{\dot{H}_1}} (\xi_{\dot{H}_1}^+ + \xi_{\dot{H}_1}^-) = \frac{1}{2} (0.031114 + 0.064752 + (-0.69685) + (-0.61703)) \\ S(\dot{H}_1) &= -0.60901 \\ S(\dot{H}_2) &= \frac{1}{\ell_{\dot{H}_2}} (\xi_{\dot{H}_2}^+ + \xi_{\dot{H}_2}^-) = \frac{1}{2} (0.242142 + 0.306855 + (-0.52531) + (-0.39811)) \\ S(\dot{H}_2) &= -0.18721 \\ S(\dot{H}_3) &= \frac{1}{\ell_{\dot{H}_3}} (\xi_{\dot{H}_3}^+ + \xi_{\dot{H}_3}^-) = \frac{1}{2} (0.949881 + 0.876593 + (-0.12341) + (-0.05012)) \\ S(\dot{H}_3) &= 0.826472 \end{aligned}$$

Now from these results it is obvious that

$$S(\dot{H}_3) > S(\dot{H}_2) > S(\dot{H}_1), \text{ so}$$

$$\begin{aligned} \dot{H}_{\sigma(1)} &= \dot{H}_3 = \{ \{0.949881, 0.876593\}, \{-0.12341, -0.05012\} \} \\ \dot{H}_{\sigma(2)} &= \dot{H}_2 = \{ \{0.242142, 0.306855\}, \{-0.52531, -0.39811\} \} \\ \dot{H}_{\sigma(3)} &= \dot{H}_1 = \{ \{0.031114, 0.064752\}, \{-0.69685, -0.61703\} \} \end{aligned}$$

Now

$$BVHFEHA(\dot{H}_1, \dot{H}_2, \dot{H}_3) = \left\{ \left\{ 0.5908, 0.5952, 0.6131, 0.6173, 0.4915, 0.4966, 0.5176, 0.5225 \right\}, \left\{ -0.3776, -0.3666, -0.3255, -0.3158, -0.2987, -0.2896, -0.2558, -0.2478 \right\} \right\}$$

Now we calculate  $\ddot{H}_1, \ddot{H}_2, \ddot{H}_3$

$$\begin{aligned} \ddot{H}_1 &= \left\{ \left\{ (0.1)^{2 \times 0.15}, (0.2)^{2 \times 0.15} \right\}, \left\{ -1 - (-(-1 - (-0.3)))^{2 \times 0.15}, -1 - (-(-1 - (-0.2)))^{2 \times 0.15} \right\} \right\} \\ \ddot{H}_1 &= \{ \{0.501187, 0.617034\}, \{-0.10148, -0.06475\} \} \\ \ddot{H}_2 &= \left\{ \left\{ (0.5)^{2 \times 0.2}, (0.6)^{2 \times 0.2} \right\}, \left\{ -1 - (-(-1 - (-0.2)))^{2 \times 0.2}, -1 - (-(-1 - (-0.1)))^{2 \times 0.2} \right\} \right\} \\ \ddot{H}_2 &= \{ \{0.757858, 0.815193\}, \{-0.08539, -0.04127\} \} \\ \ddot{H}_3 &= \left\{ \left\{ (0.9)^{2 \times 0.65}, (0.8)^{2 \times 0.65} \right\}, \left\{ -1 - (-(-1 - (-0.2)))^{2 \times 0.65}, -1 - (-(-1 - (-0.1)))^{2 \times 0.65} \right\} \right\} \\ \ddot{H}_3 &= \{ \{0.871998, 0.748199\}, \{-0.2518, -0.128\} \} \end{aligned}$$

The score values are:

$$\begin{aligned} S(\ddot{H}_1) &= \frac{1}{\ell_{\ddot{H}_1}} (\xi_{\ddot{H}_1}^+ + \xi_{\ddot{H}_1}^-) = \frac{1}{2} (0.501187 + 0.617034 + (-0.10148) + (-0.06475)) \\ S(\ddot{H}_1) &= 0.475996 \\ S(\ddot{H}_2) &= \frac{1}{\ell_{\ddot{H}_2}} (\xi_{\ddot{H}_2}^+ + \xi_{\ddot{H}_2}^-) = \frac{1}{2} (0.757858 + 0.815193 + (-0.08539) + (-0.04127)) \\ S(\ddot{H}_2) &= 0.723196 \\ S(\ddot{H}_3) &= \frac{1}{\ell_{\ddot{H}_3}} (\xi_{\ddot{H}_3}^+ + \xi_{\ddot{H}_3}^-) = \frac{1}{2} (0.871998 + 0.748199 + (-0.2518) + (-0.128)) \\ S(\ddot{H}_3) &= 0.620199 \end{aligned}$$

Now from these results it is obvious that



$S(\ddot{H}_2) > S(\ddot{H}_3) > S(\ddot{H}_1)$ , so

$$\ddot{H}_{\sigma(1)} = \ddot{H}_2 = \{\{0.757858, 0.815193\}, \{-0.08539, -0.04127\}\}$$

$$\ddot{H}_{\sigma(2)} = \ddot{H}_3 = \{\{0.871998, 0.748199\}, \{-0.2518, -0.128\}\}$$

$$\ddot{H}_{\sigma(3)} = \ddot{H}_1 = \{\{0.501187, 0.617034\}, \{-0.10148, -0.06475\}\}$$

$$BVHFEHG(\ddot{H}_1, \ddot{H}_2, \ddot{H}_3) = \left\{ \begin{array}{l} \{0.7569, 0.7843, 0.6974, 0.7239, 0.7740, 0.8015, 0.7139, 0.7406\}, \\ \{-0.2059, -0.1949, -0.1234, -0.1141, -0.1855, -0.1753, -0.1065, -0.0977\} \end{array} \right\}$$

### Application

In this section, we applied Einstein operation for BVHFS on a DM problem. First, we construct a DM matrix in which decision makers characterized every element by a BVHFE. The brief decision making process is as follows:

Consider that we have  $n$  substitutes  $A_i$  with  $m$  attributes  $x_j$  where  $(i, j=1, 2, 3, \dots, m)$  and assume that  $w = (w_1, w_2, \dots, w_m)$  be the weight vector such that  $w_j \in [-1, 1]$ ,  $j=1, 2, 3, \dots, m$  and  $\sum_{j=1}^m w_j = 1$ . The analysts take values for the substitutes  $A_i$  under the attributes  $x_j$  in the form of BVHFEs  $(H_{ij})$  anonymously.

This method is illustrated as follows:

Step 1:

In this step decision matrix is established.

Step 2:

In this step, BVHFE  $\alpha_i (i=1, 2, 3, \dots, n)$  can be obtained for the substitutes  $A_j (j=1, 2, 3, \dots, n)$  by defined aggregation operators.

Step 3:

In this step, score values are calculated.

Step 4:

Based on score analysis, we get the best alternatives.

### Example 5:

A high school needs a mathematics lecturer for their secondary classes. The school management advertised the post for a lecturer in the newspaper and several

master degree holders applied for it. The management of school will have to appoint only one candidate that is most suitable. The board for the appointment of lecturer consists of the CEO of school, the principal of school, and the director academics of school. The school is looking for a well-experienced, hard worker and a skillful subject specialist for the post described above. After the posting of advertisement in the newspaper, a number of qualified persons applied for the post. Initial screening tests were organized and three candidates were selected for interview.

Let  $A = \{A_1, A_2, A_3\}$  be the set of substitutes and  $X = \{x_1, x_2, x_3\}$  be the set of attributes and let  $w = (0.25, 0.35, 0.40)^T$  be the weight vector of the attributes  $X_i (i=1, 2, 3)$  and be  $w = (0.3, 0.5, 0.2)^T$  the aggregation associated weight vector.

Now for the selection of most suitable candidate, we use DM method.

Step 1:

Avoiding any kind of influence, the decision makers, in the state of being anonymous, construct the decision matrix shown in the table below.

**Table 1 (Decision Matrix)**

	x1	x2	x3
$A_1$	$H_{11} = \{\{0.4, 0.5\}, \{-0.3, -0.2\}\}$	$H_{12} = \{\{0.6, 0.7\}, \{-0.1, -0.2\}\}$	$H_{13} = \{\{0.3, 0.4\}, \{-0.6, -0.7\}\}$
$A_2$	$H_{21} = \{\{0.6, 0.8\}, \{-0.4, -0.6\}\}$	$H_{22} = \{\{0.4, 0.6\}, \{-0.3, -0.1\}\}$	$H_{23} = \{\{0.7, 0.9\}, \{-0.4, -0.2\}\}$
$A_3$	$H_{31} = \{\{0.3, 0.5\}, \{-0.6, -0.5\}\}$	$H_{32} = \{\{0.5, 0.8\}, \{-0.5, -0.3\}\}$	$H_{33} = \{\{0.4, 0.6\}, \{-0.1, -0.2\}\}$

**Step 2 (a):**

We now aggregate the values of given decision matrix and calculate respectively by using BVHFEWA operator.

$$H_i = BVHFEWA(H_{i1}, H_{i2}, H_{i3})$$

$$\begin{aligned} H_1 &= \left\{ \begin{array}{l} \{0.4896, 0.5530, 0.5686, 0.5178, 0.5343, 0.5786, 0.5936\}, \\ \{-0.2064, -0.2150, -0.2865, -0.2979, -0.1821, -1.898, -0.2541, -0.2644\} \end{array} \right\} \\ H_2 &= \left\{ \begin{array}{l} \{0.5322, 0.6133, 0.6218, 0.6905, 0.6137, 0.6836, 0.6909, 0.7490\}, \\ \{-0.3472, -0.3031, -0.2058, -0.1777, -0.3960, -0.3470, -0.2376, -0.2057\} \end{array} \right\} \\ H_3 &= \left\{ \begin{array}{l} \{0.4237, 0.4669, 0.6212, 0.6532, 0.4809, 0.5213, 0.6634, 0.6925\}, \\ \{-0.3981, -0.4478, -0.3062, -0.3470, -0.3746, -0.4222, -0.2872, -0.3259\} \end{array} \right\} \end{aligned}$$

**Step 3 (a):**

We calculate the score value for  $H_1, H_2, H_3$  respectively.

$$S(H_1) = 0.3057 \quad S(H_2) = 0.3719 \quad S(H_3) = 0.2018$$

**Step 4 (a):**

By ordering the score values we obtained the result as follows:

$$\text{As } S(H_2) > S(H_1) > S(H_3)$$

So  $A_2$  is the best candidate.

We now aggregate the values of given decision matrix and calculate  $H_1, H_2, H_3$  respectively by using BPVHFEWG operator.

**Step 2 (b):**

$$H_1 = \text{BPVHFEWG}(H_{11}, H_{12}, H_{13})$$

$$H_1 = \left\{ \begin{array}{l} \{0.4678, 0.4932, 0.5099, 0.5368, 0.4994, 0.5260, 0.5434, 0.5714\}, \\ \{-0.3509, -0.3950, -0.4516, -0.5017, -0.3022, -0.3445, -0.3929, -0.4400\} \end{array} \right\}$$

$$H_2 = \left\{ \begin{array}{l} \{0.5104, 0.5428, 0.6193, 0.6557, 0.5624, 0.5968, 0.6774, 0.7154\}, \\ \{-0.5381, -0.4482, -0.3279, -0.2638, -0.6959, -0.5881, -0.4423, -0.3710\} \end{array} \right\}$$

$$H_3 = \left\{ \begin{array}{l} \{0.4127, 0.4492, 0.5377, 0.5812, 0.4787, 0.5191, 0.6160, 0.6628\}, \\ \{-0.7873, -0.8752, -0.5328, -0.5881, -0.7002, -0.7769, -0.4643, -0.5140\} \end{array} \right\}$$

**Step 3 (b):**

We calculate the score value for  $H_1, H_2, H_3$  respectively.

$$\hat{S}(H_1) = 0.1211 \quad \hat{S}(H_2) = 0.1506 \quad \hat{S}(H_3) = -0.1227$$

**Step 4 (b):**

By ordering the score values we obtained the result as follows:

$$\text{As } \hat{S}(H_2) > \hat{S}(H_1) > \hat{S}(H_3)$$

So  $A_2$  is the best candidate.

So either by using BPVHFEWA operator or by BPVHFEWG operator we observe that candidate is the most suitable among three.

**CONCLUSION**

In our treatise, we successfully define Einstein aggregation operators for BVHFSs and proved some useful results. We apply the newly defined operations on a DM problem and got the same result by using two different kinds of aggregation operators. Same results could be obtained if we use Einstein ordered weighted averaging or geometric operator or Einstein hybrid operator for BVHFSs.

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